Review of Propositional Logic

1 Gold's theorem and rationalism

(1) **Gold's theorem:** Not proper superset of the class of finite languages (L_{FIN}) is learnable/identifiable.

- It holds because learners (in the Gold sense) have to be able to identify a language from **any** text. only from positive evidence.
- There are texts that 'trick' the learners into guessing a larger language rather than its subset.
- There is no negative evidence to guide the learner into the smaller language.

How does this result provide an argument in favour of innate constraints on language acquisition (such as universal grammar?).

(2) Form of the argument (when the theorem is understood correctly)

a. If there are no constraints on what languages humans can learn, then if a human learner were exposed to the correct (positive) data, then they could learn any language in $\mathcal{P}(\Sigma^*)$.
   - So human learners could be thought of as functions that can learn/identify the set of possible languages.

b. Human language learning is a process that involves reasoning from only positive data, not negative data.

c. But the set of possible languages is obviously a proper superset of the set of finite languages, so by Gold's theorem, there is a contradiction. $\perp$

Suppose we like this argument, and we conclude that there must be constraints on the set of languages that humans can learn.

- Where do these constraints come from?

(3) **Rationalist proposal:** They are innate.

Discuss alternatives to this argument:

1. Negative evidence (work by Eve Clark: actually kids do get some negative evidence).
2. Negative evidence is statistical.
3. Constraints could arise from general processes (like evolution, Jäger).

2 The syntax of propositional logic

(4) The set of **atomic formulas**

a. $\text{AF} = \{\langle P, n\rangle | n \in \mathbb{N}\}$

We usually write P_n for $\langle P, n\rangle$.

(5) Vocabulary : $\text{V} = \text{AF} \cup \{\&, \vee, \neg, (,)\}$

We define one unary function on V^*, Neg, and two binary functions And and Or.

(6) **Rules**

a. $\text{Neg}(\phi) = \neg\phi$
b. $\text{And}(\phi, \psi) = \phi \ \& \ \psi$
c. $\text{Or}(\phi, \psi) = \phi \ \vee \ \psi$

Définition 2.1 *PROP (the language of propositional logic) is the closure of AF under Neg, And and Or.*

(7) a. Set $\text{PROP}_0 = AF$ and,
b. for all natural numbers n, $PROP_{n+1} = PROP_n \cup \{Neg(\phi)|\phi \in PROP_n)\} \cup \{And(\phi, \psi)|\phi, \psi \in PROP_n\} \cup \{Or(\phi, \psi)|\phi, \psi \in PROP_n\}$.
c. Then PROP $= \{\tau \in V * |$ for some $n, \tau \in PROP_n$.
So $PROP = \bigcup_{n \in \mathbb{N}} PROP_n$.

(8) Theorems

a. $AF \subseteq PROP$
b. $PROP$ is closed under Neg, And and Or.
c. If a set K includes all the atomic formulas and is closed under Neg, And and Or, then $PROP \subseteq K$.
•PROP is the least subset of $V*$ which includes the atomic formulas and is closed under Neg, And and Or.

Some notation:

- We will use $(\phi \rightarrow \psi)$ to abbreviate $((\neg\phi) \vee \psi)$.
- AF(ϕ) is the set of atomic formulas which occur in ϕ.

(9) **Theorem:** PROP is syntactically unambiguous.

a. Each generating function Neg, And, Or is one to one,
b. The ranges of any two of NEG, AND, OR are disjoint, and
c. AF and the range of any of NEG, AND, and OR are disjoint.

How to prove theorems about PROP: an example:

Théorème 2.1 *For all $\phi \in PROP$, AF(ϕ) is finite.*

Proof: We define a set that has the particular property we're interested in, and then we show that PROP is a subset of this set.

(10) Define $K = \{\phi \in PROP | AF(\phi)$ is finite.$\}$

Now we show that K has all the atomic formulas and is closed under Neg, And and Or. So it contains PROP.

1. Since AF(P_n) $= \{P_n\}$, $P_n \in K$, for all $P_n \in AF$.
2. Suppose $\phi \in K$ to show $\neg\phi \in K$. Since $\phi \in K$, AF(ϕ) is finite. Since Neg doesn't add any atomic fomulas, $\neg\phi \in K$ also.
3. Let $\phi, \psi \in K$ to show that $\phi \vee \psi$ (and $\phi \ \& \ \psi$) $\in K$. AF($\phi \vee \psi$) = AF(ϕ) $\cup$ AF(ψ), which is finite, since the union of two finite sets is finite. ☐

2.1 Syntactic rules (proofs)

(11) We write $S \vdash \phi$ to say that there is a **proof** of ϕ from premises in S.

Crucial here is that the notion proof is purely syntactic.

Définition 2.2 *A* ***proof*** *of ϕ from premises S is a finite sequence of formulas ending in ϕ, each of which is drawn from S and marked as a premise or is derived by syntactic rule from earlier formulas in the sequence.*

There are just finitely many rules.

(12) Examples of syntactic rules

a. **Conjunction Elimination**: If $(\psi\&\chi)$ is a line in a proof then we can add ψ to the end; also we can add χ to the end.

b. **Modus Ponens:** if $(\psi \to \chi)$ are both lines in the proof then we can add χ to the proof.

3 Semantics of propositional logic

Définition 3.1 *A* ***model*** *for PROP is a function $v : AF \to \{1, 0\}$.*

- *v is often called a valuation.*

Définition 3.2 *For each model v we define a function v^* from PROP into {1, 0} by setting:*

(13) a. $v^*(P_n) = v(P_n)$, *for all atomic formulas p_n,*

b. $v^*(\neg\phi) = 1$ *iff* $v^*\phi) = 0$,

c. $v^*(\phi\ \&\ \psi) = 1$ *iff* $v^*(\phi) = v^*(\psi) = 1$, *and*

d. $v^*(\phi \vee \psi) = 0$ *iff* $v^*(\phi) = v^*(\psi) = 0$.

v^* is called an interpretation of PROP.

- Our definition of interpretation is fully compositional: The interpretation (truth value) of a complex formula is uniquely determined by the interpretations (truth values) of the formulas it is built from.

- The truth of a formula depends only on the truth of the atomic formulas which occur in it.

Théorème 3.1 ***The coincidence lemma.*** *For all $\phi \in PROP$ and all models v and u, if $v(P_n) = u(P_n)$ for all atomic formulas occurring in ϕ, then $v^*(\phi) = u^*(\phi)$.*

4 Entailment

(14) a. For $\phi, \psi \in PROP$, $\phi \vDash \psi$ (read: ϕ **entails/logically implies** ψ) iff for all models v for PROP, if $v^*(\phi) = 1$ then $v^*(\psi) = 1$.

b. For all $K \subseteq PROP$, all $\phi \in PROP$, $K \vdash \phi$ iff for all models v, if $v^*(\tau) = 1$ for all $\tau \in K$, then $v^*(\phi) = 1$.

(15) For $\phi \in PROP$, ϕ is logically true (valid, a tautology) iff for all models v, $v^*(\phi) = 1$. To say that ϕ is logically true, we write $\vDash \phi$.

(16) For $\phi, \psi \in PROP$, ϕ is logically equivalent to ψ, noted $\phi \equiv \psi$, iff for all models $v, v^*(\phi) = v^*(\psi)$.

(17) Some theorems

a. $\phi \equiv \psi$ iff $\phi \models \psi$ and $\psi \models \phi$

b. $\phi \equiv \psi$ iff $\models (\phi \leftrightarrow \psi)$

c. $\phi \models \psi$ iff $\models (\phi \rightarrow \psi)$

Thus, when we want to show equivalence between two notions, we can prove the biconditional statement. Likewise, if we want to show that ψ is a consequence of ϕ, we assume ϕ to show ψ is the case.

5 Decidability

PROP has the pleasing property that there is a general mechanical procedure (an algorithm) for deciding whether an arbitrary formula ϕ is logically true or not.

- The procedure is called a Decision Procedure and PROP is said to be decidable.
- Similarly there is a procedure for deciding whether arbitrary formulas ϕ, ψ are logically equivalent, or whether one entails the other.

Consider the formula $((P_2 \& P_5) \vee (\neg P_2 \& \neg P_5))$.

- To test whether it is logically true we must evaluate its truth under all interpretations.
- But by the Coincidence Lemma we need only consider a model v in so far as it assigns truth values to P2 and P5, the atomic formulas occurring in it.
- Now there are just two ways we can assign truth values to P_2, and for each of those there are two ways to assign truth values to P_5.

So let us list all cases, writing under each atomic formula the value we assign it and then computing the truth value of the entire formula by writing the truth value of a derived formula under the connective $(\&, \vee, \neg)$ used to build it.

P_2	P_5	$((P_2 \& P_5)$	$\vee$	$(\neg P_2$	$\&$	$\neg P_5))$
1	1	1	1	0	0	0
1	0	0	0	0	0	1
0	1	0	0	1	0	0
0	0	0	1	1	1	1

Generalizing, since any formula in SL is built from just finitely many atomic formulas (Theorem 6.5) we can decide the validity of any such formula by con- structing a truth table.

- Similarly to show that two formulas are not logically equivalent it suffices to illustrate one line of their truth table in which they have different values.
- If they have the same value for all lines then they are logically equivalent.
- To show that some ϕ entails some ϕ you must show that for each line of the truth table for ϕ which makes it true, ψ is also true in that case.
- To falsify the entailment claim it suffices to find one assignment of truth values to the atomic formulas of ϕ which make ϕ true but ψ false.

This decidability result should not however go to our heads.

- Given n atomic formulas, since each one is two valued the number of "possible combinations" above is 2^n.
- So the number of lines in the truth table quickly gets too large to realistically compute and the problem is said to be **intractable**.
- Ex. if you have 100 AFs, the number of lines in the truth table is 2^{100}: a 31 digit number, much greater than the number of seconds since the Bing Bang (Keenan & Moss 2017, Harel 1987).

But we have useful theorems that can help us make things more tractable:

Théorème 5.1 ***Compactness.*** *For $S \subseteq PROP$ and $\phi \in PROP$, $S \vDash \phi$ iff there is a finite subset K of S such that $K \vDash \phi$.*

Compactness tells us that in PROP the truth of a claim ϕ cannot depend on infinitely many premises. If ϕ follows from some infinite set then there is a finite subset from which it follows.

Théorème 5.2 ***Interpolation.*** *For ϕ, ψ non-trivial (neither true nor false in all models), if $\phi \vDash \psi$ then there is a $\tau \in PROP$ such that $\phi \vDash \tau$ and $\tau \vDash \psi$ and $AF(\tau) \subseteq AF(\phi) \cap AF(\psi)$.*

So if a formula non-trivially entails another that fact just depends on the interpretations of the atomic formulas they have in common.

6 Syntax and semantics: Soundness and completeness

Connections between syntax and semantics:

Théorème 6.1 ***Soundness.*** *For all $S \subseteq PROP$, all $\phi \in PROP$, if $S \vdash \phi$ then $S \vDash \phi$.*

Théorème 6.2 ***Completeness.*** *For all $S \subseteq PROP$, all $\phi \in PROP$, if $S \vDash \phi$ then $S \vdash \phi$.*

The completeness property of SL tells us that in SL we can syntactically characterize the entailment relation.