Intensional Propositional Logic

1 Overview

Intensional propositional logic emerged in the 1950s with the work of Carnap, Kanger, Hintikka and Kripke.

- It was developed in part to give a formal semantic analysis of sentences in natural language which cannot be analyzed with simple propositional logic.
- Enormous uses in other parts of natural language semantics and outside linguistics (epistemology).

(1) a. It is necessary that Sara come to the party.
b. It is possible for Sara to come to the party.
c. Sarah must come to the party.
d. Sarah can come to the party.

We extend out language PROP with **operators** (Os), to make Intensional propositional logic (IPROP).

- For all $\phi \in PROP$, $O\phi \in IPROP$

(2) Examples
a. OP_{23}
b. $OP_2 \rightarrow OP_2$
c. $O(OP_{83} \& OP_{24})$

In natural language, we many intensional operators (to name a few):

(3) *it ought to be the case that, I know that, it will always be the case that, it was once the case that, it is necessary that, it is possible that...*

If our operator stands for *I know that...*, (2-a) symbolizes *I know that* P_{23}; (2-b) symbolizes *If I know that* P_2*, I know that* P_2; and (2-c) symbolizes *I know that I know that* P_{83} *and I know that* P_{24}.

The semantics of intensional operators needs to take into account *contexts*:

- For temporal operators, like *it will always be the case that* and *it was once the case that*, these contexts are moments in time.
- For modal operators, like *it is necessary that* and *it is possible that*, the contexts are **possible situations** (possible worlds).

Intensional logic has a **context dependent** notion of meaning: which truth values a proposition has are not absolute, but are relative to the contexts in which their truth is evaluated.

- Formally: we replace the simple semantics in which formulas receive absolute truth values with a system in which valuation functions assign truth values only relative to some context k (taken from the set K of such contexts).

(4) a. *it was once the case that* ϕ is true **at** k just in case there is some context (point in time) k' earlier than the present context (point in time) k at which ϕ was true.
b. *I know that* ϕ is true in a context k just in case ϕ is true in k but also that ϕ is true in all contexts k' which are compatible with the knowledge I have in k (called *epistemic alternatives*).

(5) Different kinds of necessity
a. *It is logically necessity that* ϕ is true in any context k just in case ϕ is true in every possible context k'.
b. *It is a physical necessaity that* ϕ is true in any context k just in case ϕ is true in those contexts k' where the same physical laws hold as in k itself.

What contexts must be taken into account in evaluating a formula $O\phi$ may depend not only on the intended interpretation of O but also on the particular context in which the evaluation is to take place.

- Those contexts k' which are relevant when evaluating within a context k are said to be **accessible** from k.

(6) a. *It is necessary that* ϕ is true at k iff ϕ is true in **all** contexts accessible from k.
b. *It is possible that* ϕ is true at k iff ϕ is true in **some** contexts accessible from k.

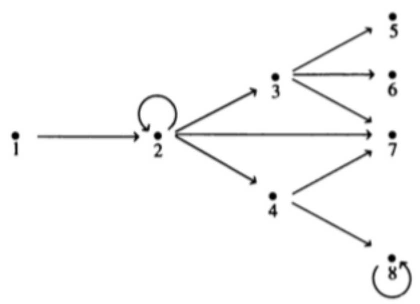
2 Syntax and Semantics

Syntax: If ϕ is a formula in IPROP, then $\Box\phi$ and $\Diamond\phi$ are too.

Définition 2.1 *A model M for IPROP consists of:*

1. *A nonempty set W of possible worlds.*
2. *A binary relation R on W, the accessibility relation.*
3. *A valuation function v which assigns a truth value $v_w(P)$ to every atomic proposition P in each world $w \in W$.*

(1)



- Sometimes a special character w_0 of W is singled out as the actual world.
- A set of possible worlds W together with a suitable accessibility relation R is referred to as a *frame* or *structure.*
- A model consists of a frame F together with a valuation function v.
  - Frames fix which possible worlds there are and which are accessible from which others.
  - Valuations decide which facts obtain in each of the possible worlds.

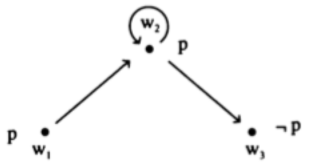
Définition 2.2 ***Truth definition.*** *If M is a model with W as its set of possible worlds, R as its accessibility relation, and v as its valuation, then $v_{M,w}(\phi)$ is defined by the following clauses:*

1. $v_{M,w}(P) = v_w(P)$, *for all atomic propositions* P.
2. $v_{M,w}(\neg\phi) = 1$ *iff* $v_{M,w}(\phi) = 0$.
3. $v_{M,w}(\phi \to \psi) = 1$ *iff* $v_{M,w}(\phi) = 0$ *or* $v_{M,w}(\psi) = 1$
4. *Similarly for & and* $\vee$ ...
5. $v_{M,w}(\Box\phi) = 1$ *iff for all* $w' \in W$ *such that* wRw', $v_{M,w'}(\phi) = 1$
6. $v_{M,w}(\Diamond\phi) = 1$ *iff for at least one* $w' \in W$ *such that* wRw', $v_{M,w'}(\phi) = 1$

- Sometimes if the model is clear, we leave off the M diacritic.
- Note the analogy with $\forall$ and $\exists$.

Examples:

(9)



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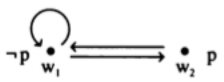


- Write out the Rs as ordered pairs.

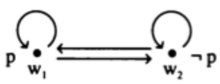
Suppose we have a valuation function v such that $v_{w_1}(p) = v_{w_2}(p) = 1$ and $v_{w_3}(p) = 0$.

- What are the truth values of $\Box p$ and $\Diamond p$ at the different worlds? $\Diamond\neg p$? $\Box\Diamond p$? $\Diamond\Box\neg p$?

We say that formulas that are true in each of a model's worlds are **valid** in the model (written $v_M(\phi) = 1$).

- We distinguish between validities that depend on a valuation ($\Diamond p \& \Diamond\neg p$) and those that do not ($\Box p \to p$).

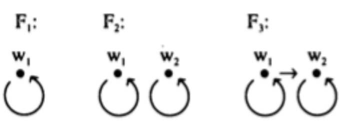
(11)



If a formula ϕ is valid on the models constructed on the basis of a frame F, then we say it is *valid* on F.

- Validity of such formulas tend to express a property of a class of frames.

(13)



- $\Box\phi \to \phi$ is valid on just those frames with **reflexive** accessability relations: it characterizes the class of reflexive frames.
- A relation R on a set A is reflexive iff for all $a \in A$, aRa.

Théorème 2.1 *$\Box p \rightarrow p$ is valid on F iff F has a reflexive accessibility relation.*

Proof $\Leftarrow$ Suppose F has a reflexive accessibility relation. Let $w \in W$ to show $v_w(\Box p \rightarrow p) = 1$. Suppose that $v_w(\Box p) = 1$. Then, for $w' \in W$ accessible from v_w (wRw'), $v'_w(p) = 1$. Since F is accessible, wRw, so $v_w(p) = 1$.

$\Rightarrow$ Suppose $\Box p \rightarrow p$ is valid on F, for all atomic propositions p. Suppose, for a contradiction that F is a frame whose accessibility relation is not reflexive, i.e. there is some F such that we do not have wRw, for some $w \in W$. Consider a valuation v such that $v_w(p) = 0$ and for all other $w' \in W$, $v_{w'}(p) = 1$ (for some atomic proposition p). So $v_w(\Box p) = 1$ and $v_w(p) = 0$. So $v_w(\Box p \rightarrow p) = 0$ meaning that this formula is not valid on F. $\perp$ So F has a reflexive accessibility relation. $\Box$

(7) Other formulas that characterize accessibility relations:

a. $\Box \phi \rightarrow \Box\Box \phi \mapsto$ transitivity

b. $\Diamond\Box \phi \rightarrow \phi \mapsto$ symmetry